Sequences of functions

Setup: (X,ρ), (Y,d) - metric spaces.
$C(X,Y) := \{f: X \to Y, f \text{ - continuous}\}$.
$\mathcal{F}(X,Y) \coloneqq \{f: X \to Y\}$.

Definition.
Let (f_n) be a sequence, $f_n \in \mathcal{F}(X,Y)$, and $f \in \mathcal{F}(X,Y)$.
We say that f_n converges to f *pointwise* to f, if $\forall x \epsilon X$ the sequence $f_n(x)$ converges to $f(x)$. In other words, $\forall x \in X, \lim_{n\to\infty} d(f_n(x), f(x)) = 0$.
We say that f_n converges to f uniformly if $\lim_{n\to\infty} \sup_{x\in X} d(f_n(x), f(x)) = 0$.
Remark.
If $Y = \mathbb{R}$ with usual metric, then uniform convergence is the same as convergence in $\mathcal{F}(X)$!
Important example.
$f_n(x) \coloneqq x^n$
For any $a < 1$, it converges uniformly on [0,a] to 0.
But on [0,1] this sequence converges pointwise, but not uniformly, to the following function:
$f(x) = \begin{cases} 0, 0 \le x < 1 \\ 1, x = 1 \end{cases}$
Message: a sequence of continuous functions can converge pointwise to a discontinuous function. But not uniformly.
Theorem.
Assume $f_n(x)$ converge to $f(x)$ uniformly, and each f_n is continuous. Then f is also continuous.
Proof.
Let $x \in X$. We need to prove that f is continuous at x.
For this, let us fix $\varepsilon > 0$. Let us find n such that $\forall z \in X$: $d(f_n(z), f(z)) < \varepsilon/3$ (we use uniform convergence here!).
The function f_n is continuous at x. Thus $\exists \delta > 0: \rho(x,z) < \delta \Rightarrow d(f_n(x), f_n(z)) < \frac{\varepsilon}{3}$.
Now, when $\rho(x,z) < \delta$, we have

$$d\big(f(x), f(z)\big) \le d(f_n(z), f(z)) + d(f_n(x), f_n(z)) + d(f_n(x), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \blacksquare$$

Definition.
$C_b(X) \coloneqq \{f: X \to \mathbb{R}, f \text{ - continuous and bounded}\}$ with the uniform metric ρ.

Remark.
If X is a compact metric space, the condition of boundedness is unnecessary. Notation in this case: $C(X)$.
Corollary.
$C_b(X)$ is complete.
Proof.
The previous theorem means that $C_b(X)$ is a closed subspace of complete metric space $\mathcal{F}(X)$ ■

Series of functions

We now consider functions from a metric space (X, ρ) to $\mathbb{R}$.

Definition.
We say that the series
$\sum_{n=1}^{\infty} f_n(x)$ converges uniformly(pointwise) if the sequence of partial sums
$s_k(x) := \sum_{n=1}^{k} f_n(x)$ converges uniformly(pointwise).

M-test.
Let $\forall x \in X\ |f_n(x)| \leq M_n$ and $\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.
Proof.
Since $C_b(X)$ is complete, it is enough to show that $s_k(x)$ form a Cauchy sequence.
Let us fix $\varepsilon > 0$ and pick $N: m > N \Rightarrow \sum_{n=m}^{\infty} M_n < \varepsilon$.
It implies that $\forall k > m > N$and for any $x \in X$ we have
$$|s_k(x) - s_m(x)| = \left|\sum_{n=m+1}^{k} M_n\right| \leq \left|\sum_{n=m+1}^{\infty} M_n\right| < \varepsilon \blacksquare$$
Remark.
It is not a necessary condition! Consider $f_n(x) \equiv {(-1)^n}/{n}$.

Important example: power series
Let (a_n) be a sequence of real numbers. The *power series with coefficients* (a_n) is defined as
$\sum_{n=1}^{\infty} a_n x^n$.

With each power series we associate its *radius of convergence*
$R = R((a_n)) := 1/\limsup_{n\to\infty} \sqrt[n]{|a_n|}$.
Here we assume that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.
Theorem.
If $|x| > R$ the series $\sum_{n=1}^{\infty} a_n x^n$ diverges.
If $r < R$ the series $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly in the interval $[-r, r]$.
Proof.
Root test.$\blacksquare$